

On (strong) proper vertex-connection of graphs*

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Abstract

A path in a vertex-colored graph is a *vertex-proper path* if any two internal adjacent vertices differ in color. A vertex-colored graph is *proper vertex k -connected* if any two vertices of the graph are connected by k disjoint vertex-proper paths of the graph. For a k -connected graph G , the *proper vertex k -connection number* of G , denoted by $pvc_k(G)$, is defined as the smallest number of colors required to make G proper vertex k -connected. A vertex-colored graph is *strong proper vertex-connected*, if for any two vertices u, v of the graph, there exists a vertex-proper u - v geodesic. For a connected graph G , the *strong proper vertex-connection number* of G , denoted by $spvc(G)$, is the smallest number of colors required to make G strong proper vertex-connected. These concepts are inspired by the concepts of rainbow vertex k -connection number $rvck(G)$, strong rainbow vertex-connection number $srvc(G)$, and proper k -connection number $pc_k(G)$ of a k -connected graph G . Firstly, we determine the value of $pvc(G)$ for general graphs and $pvc_k(G)$ for some specific graphs. We also compare the values of $pvc_k(G)$ and $pc_k(G)$. Then, sharp bounds of $spvc(G)$ are given for a connected graph G of order n , that is, $0 \leq spvc(G) \leq n - 2$. Moreover, we characterize the graphs of order n such that $spvc(G) = n - 2, n - 3$, respectively. Finally, we study the relationship among the three vertex-coloring parameters, namely, $spvc(G)$, $srvc(G)$ and the chromatic number $\chi(G)$ of a connected graph G .

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1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to book [2] for undefined notation and terminology in graph theory. For simplicity, a set of internally vertex-disjoint paths will be called *disjoint*. A path in an edge-colored graph is a *rainbow path* if its edges have different colors. An edge-colored graph is *rainbow k -connected* if any two vertices of the graph are connected by k disjoint rainbow paths of the graph. For a k -connected graph G , the *rainbow k -connection number* of G , denoted by $rc_k(G)$, is defined as the smallest number of colors required to make G rainbow k -connected. These concepts were first introduced by Chartrand et al. in [4, 5]. Since then, a lot of results on the rainbow connection have been obtained; see [12, 13].

As a natural counterpart of the concept of rainbow k -connection, the concept of rainbow vertex k -connection was first introduced by Krivelevich and Yuster in [8] for $k = 1$, and then by Liu et al. in [14] for general k . A path in a vertex-colored graph is a *vertex-rainbow path* if its internal vertices have different colors. A vertex-colored graph is *rainbow vertex k -connected* if any two vertices of the graph are connected by k disjoint vertex-rainbow paths of the graph. For a k -connected graph G , the *rainbow vertex k -connection number* of G , denoted by $rvc_k(G)$, is defined as the smallest number of colors required to make G rainbow vertex k -connected. There are many results on this topic, we refer to [6, 7, 11].

In 2011, Borozan et al. [3] introduced the concept of proper k -connection of graphs. A path in an edge-colored graph is a *proper path* if any two adjacent edges differ in color. An edge-colored graph is *proper k -connected* if any two vertices of the graph are connected by k disjoint proper paths of the graph. For a k -connected graph G , the *proper k -connection number* of G , denoted by $pc_k(G)$, is defined as the smallest number of colors required to make G proper k -connected. Note that

$$1 \leq pc_k(G) \leq \min\{\chi'(G), rc_k(G)\}, \quad (1)$$

where $\chi'(G)$ denotes the edge-chromatic number. Recently, the case for $k = 1$ has been studied by Andrews et al. [1] and Laforge et al. [9].

Inspired by the concepts above, now we introduce the concept of proper vertex k -connection. A path in a vertex-colored graph is a *vertex-proper path* if any two internal adjacent vertices differ in color. A vertex-colored graph is *proper vertex k -connected* if any two vertices of the graph are connected by k disjoint vertex-proper paths of the graph. For a k -connected graph G , the *proper vertex k -connection number* of G , denoted by $pvc_k(G)$, is defined as the smallest number of colors required to make G proper vertex k -connected. We write $pvc(G)$ for $pvc_1(G)$, and similarly, $rc(G)$, $rvc(G)$ and $pc(G)$ for $rc_1(G)$, $rvc_1(G)$

and $pc_1(G)$. Note that

$$0 \leq pvc_k(G) \leq \min\{\chi(G), rvc_k(G)\}, \quad (2)$$

where $\chi(G)$ denotes the chromatic number of G . By Brooks' theorem [2], if G is a connected graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$, and so $pvc_k(G) \leq \Delta$, where Δ denotes the maximum degree of G .

2 Proper vertex k -connection

In this section, we determine the value of $pvc(G)$ for general graphs and $pvc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. Moreover, we show that $pc_k(G) \geq pvc_k(G)$ for $k = 1$ and provide an example graph G such that $pc_k(G) > pvc_k(G)$ for $k \geq 2$.

2.1 Proper vertex-connection number $pvc(G)$

From the definition of $pvc(G)$, the following results are immediate. Recall that the *diameter* of a connected graph G , denoted by $diam(G)$, is the maximum of the distances among pairs of vertices of G .

Proposition 1. *Let G be a nontrivial connected graph. Then*

- (a) $pvc(G) = 0$ if and only if G is a complete graph;
- (b) $pvc(G) = 1$ if and only if $diam(G) = 2$.

For the case that $diam(G) \geq 3$, we have the following theorem.

Theorem 1. *Let G be a nontrivial connected graph. Then, $pvc(G) = 2$ if and only if $diam(G) \geq 3$.*

Proof. The necessity can be verified by Proposition 1.

Now we prove its sufficiency. Since $diam(G) \geq 3$, we have that $pvc(G) \geq 2$ and then we just need to prove that $pvc(G) \leq 2$. Let T be a spanning tree of G . For a vertex $v \in V(T)$, let $e_T(v)$ denote the eccentricity of v in T , i.e., the maximum of the distances between v and the other vertices in T . Let $V_i = \{u \in V(T) : d_T(u, v) = i\}$, where $0 \leq i \leq e_T(v)$. Hence $V_0 = \{v\}$. Define a 2-coloring of the vertices of T as follows: If i is odd, color the vertices of V_i with color 1; otherwise, color the vertices of V_i with color 2. It is easy to check that for any two vertices x and y in G , there is a vertex-proper path connecting them. Thus, $pvc(G) \leq 2$, and therefore, $pvc(G) = 2$. \square

2.2 Proper vertex k -connection of some specific graphs

In this subsection, we shall determine the value of $pvc_k(G)$ for some specific graphs. Let $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$ denote the vertex-connectivity of G . Note that $pvc_k(G)$ is well defined if and only if $1 \leq k \leq \kappa(G)$. We start with the case that G is a cycle of order n , denoted by C_n . Observe that $\kappa(C_n) = 2$. We have the following results.

Theorem 2.

- (a) $pvc(C_3) = 0$, $pvc(C_4) = pvc(C_5) = 1$, and $pvc(C_n) = 2$ for $n \geq 6$.
- (b) $pvc_2(C_3) = 1$, $pvc_2(C_n) = 2$ for $n \geq 4$ even, and $pvc_2(C_n) = 3$ for $n \geq 5$ odd.

Proof. (a) Since $C_3 = K_3$ and $diam(C_4) = diam(C_5) = 2$, we have that $pvc(C_3) = 0$ and $pvc(C_4) = pvc(C_5) = 1$ by Proposition 1. Since $diam(C_n) \geq 3$ for $n \geq 6$, it follows that $pvc(C_n) = 2$ ($n \geq 6$) by Theorem 1.

(b) The assertion can be easily verified for C_3 . Now, let $n \geq 4$. We consider two cases, depending on the parity of n .

Case 1. n is even. By (2), we have that $pvc_2(C_n) \leq \chi(C_n) = 2$. If one colors the vertices of C_n with one color, then we do not have two vertex-proper paths between any two adjacent vertices. Hence, $pvc_2(C_n) = 2$ for $n \geq 4$ even.

Case 2. n is odd. Similarly from (2), it follows that $pvc_2(C_n) \leq \chi(C_n) = 3$. Assume that $C_n = v_1v_2 \cdots v_nv_1$ ($n \geq 5$). If we have a vertex-coloring for C_n with two colors, then there must exist two adjacent vertices, say v_1 and v_2 , colored the same. However, there do not have two vertex-proper paths between v_n and v_3 . Thus, $pvc_2(C_n) = 3$ for $n \geq 5$ odd. \square

A graph obtained from C_n by joining a new vertex v to every vertex of C_n is the *wheel* W_n . The vertex v is the center of W_n . Note that $\kappa(W_n) = 3$.

Theorem 3.

- (a) $pvc(W_3) = 0$ and $pvc(W_n) = 1$ for $n \geq 4$.
- (b) $pvc_2(W_3) = 1$ and $pvc_2(W_n) = pvc(C_n)$ for $n \geq 4$.
- (c) $pvc_3(W_3) = 1$ and $pvc_3(W_n) = pvc_2(C_n)$ for $n \geq 4$.

Proof. (a) Since $W_3 = K_4$ and $diam(W_n) = 2$ for $n \geq 4$, we have that $pvc(W_3) = 0$ and $pvc(W_n) = 1$ ($n \geq 4$) by Proposition 1.

(b) The assertion can be easily verified for W_3 . Now, let $n \geq 4$. Take a proper vertex connected coloring for the cycle C_n in W_n with $pvc(C_n)$ colors and then color the center with any used color. Clearly, W_n is proper vertex 2-connected. Thus, $pvc_2(W_n) \leq pvc(C_n)$. On the other hand, consider a vertex-coloring for W_n with fewer than $pvc(C_n)$ colors.

Then, there exist two vertices u, v in the cycle C_n of W_n such that we do not have a vertex-proper $u-v$ path along the cycle. Hence, there is at most one vertex-proper $u-v$ path in W_n (using the center of W_n). Thus, $pvc_2(W_n) \geq pvc(C_n)$.

(c) This can be proved by a similar way as the proof of Theorem 3(b). \square

For the complete graph K_n , we have that $pvc(K_n) = 0$ and $pvc_2(K_n) = pvc_3(K_n) = \dots = pvc_{n-1}(K_n) = 1$. Let K_{n_1, n_2} denote the complete bipartite graph, where $2 \leq n_1 \leq n_2$. Clearly, $\kappa(K_{n_1, n_2}) = n_1$. Then, we have $pvc(K_{n_1, n_2}) = 1$ and $pvc_k(K_{n_1, n_2}) = 2$ for $2 \leq k \leq n_1$. Let $G = K_{n_1, \dots, n_t}$ be a complete multipartite graph, where $1 \leq n_1 \leq \dots \leq n_t$ with $t \geq 3$ and $n_t \geq 2$. In [14], Liu, Mestre and Sousa determined the rainbow vertex k -connection number of K_{n_1, \dots, n_t} . By the same method as the proof of Theorem 4 in [14] and the fact that $pvc_k(G) \leq rvc_k(G)$, we deduce the following result.

Theorem 4. *Let $1 \leq n_1 \leq \dots \leq n_t$, where $t \geq 3, n_t \geq 2$ and $m = \sum_{i=1}^{t-1} n_i$.*

(a) *If $1 \leq k \leq m - 2$, then we have the following:*

(i) $pvc_k(K_{n_1, \dots, n_t}) = 1$ if $1 \leq k \leq m - n_{t-1} + 1$.

(ii) $pvc_k(K_{n_1, \dots, n_t}) = 2$ if $m - n_{t-1} + 2 \leq k \leq m - 2$.

(b) (i) $pvc_{m-1}(K_{n_1, \dots, n_t}) = 1$ if $n_{t-1} \leq 2$.

(ii) $pvc_{m-1}(K_{n_1, \dots, n_t}) = 2$ if $n_{t-1} \geq 3$ and we do not have $n_t = n_{t-1} = n_{t-2}$ odd.

(iii) $pvc_{m-1}(K_{n_1, \dots, n_t}) = 3$ if $n_t = n_{t-1} = n_{t-2} \geq 3$ are odd.

(c) (i) $pvc_m(K_{n_1, \dots, n_t}) = 1$ if $n_{t-1} = 1$.

(ii) $pvc_m(K_{n_1, \dots, n_t}) = 2$ if $2 \leq n_{t-1} \leq n_t - 2$.

(iii) $pvc_m(K_{n_1, \dots, n_t}) = 2$ if $n_{t-1} = n_t - 1 \geq 2$ and $n_{t-2} \leq 2$, or $n_{t-1} = n_t \geq 2$ and $n_{t-2} = 1$.

(iv) $pvc_m(K_{n_1, \dots, n_t}) = 3$ if $n_{t-1} = n_t - 1$ and $n_{t-2} \geq 3$, or $n_{t-1} = n_t \geq 3, n_{t-2} \geq 2$ and we do not have $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$ and $t \geq 4$.

(v) $pvc_m(K_{n_1, \dots, n_t}) = 4$ if $t \geq 4$ and $n_t = n_{t-1} = n_{t-2} = n_{t-3} = 4$.

(vi) $pvc_m(K_{n_1, \dots, n_t}) = s$ if $n_t = n_{t-1} = \dots = n_{t-s+1} = 2$ and $n_{t-s} = n_{t-s-1} = \dots = n_1 = 1$, for $1 \leq s \leq t$.

2.3 Comparing $pc_k(G)$ and $pvc_k(G)$

In [8], Krivelevich and Yuster compared the values of $rc(G)$ and $rvc(G)$. They observed that one of $rc(G)$ and $rvc(G)$ cannot be bounded in terms of the other, by providing example graphs G where $rc(G)$ is much larger than $rvc(G)$, and vice versa. In [14], Liu et al. compared the values of $rc_k(G)$ and $rvc_k(G)$, similarly.

Here, we will compare the values of $pc_k(G)$ and $pvc_k(G)$. Note that $pc(G) = 1$ if and only if G is a complete graph. In addition, by Proposition 1 we have the following assertion: if $diam(G) = 1$, then $pc(G) = 1$ and $pvc(G) = 0$; if $diam(G) = 2$, then $pc(G) \geq 2$ and $pvc(G) = 1$; if $diam(G) \geq 3$, then $pc(G) \geq 2$ and $pvc(G) = 2$. Thus, we have $pc(G) \geq pvc(G)$. For $k \geq 2$, the following theorem shows that there exists an example graph G such that $pc_k(G) > pvc_k(G)$.

Theorem 5. *Let G be a complete bipartite graph with classes U and V , where $U = \{u_1, \dots, u_t\}$ and $V = \{v_1, \dots, v_k\}$ ($2 \leq k < t$). Then, $pc_k(G) = t$ and $pvc_k(G) = 2$.*

Proof. Clearly, $pvc_k(G) = 2$. Next we just need to prove $pc_k(G) = t$. Since G is a complete bipartite graph, it follows that $\chi'(G) = \Delta$. Moreover, $pc_k(G) \leq \chi'(G)$ in (1) and $\Delta = t$. Then, $pc_k(G) \leq t$. If one colors the edges of G with fewer than t colors, then there exist two edges of $\{v_1u_i : u_i \in U\}$, say v_1u_1 and v_1u_2 , colored the same. Thus, we can not have k disjoint proper paths between u_1 and u_2 . Therefore, $pc_k(G) = t$. \square

We observe that $pc_k(G) \geq pvc_k(G)$ for $k = 1$. Moreover from Theorem 5, we find an example such that $pc_k(G) > pvc_k(G)$ for $k \geq 2$. Note that $pc_2(G) = pvc_2(G)$ if G is a cycle of order $n \geq 4$. However, we cannot show whether there exists a graph G such that $pc_k(G) < pvc_k(G)$. Thus, we pose the following problem.

Problem 1. *Let $k \geq 2$. Does it hold that $pc_k(G) \geq pvc_k(G)$ for any connected graph G ?*

3 Strong proper vertex-connection

In [10], Li et al. introduced the concept of strong rainbow vertex-connection. A vertex-colored graph is *strong rainbow vertex-connected*, if for any two vertices u, v of the graph, there exists a vertex-rainbow u - v geodesic, i.e., a u - v path of length $d(u, v)$. For a connected graph G , the *strong rainbow vertex-connection number* of G , denoted by $srvc(G)$, is the smallest number of colors required to make G strong rainbow vertex-connected.

A natural idea is to introduce the concept of the strong proper vertex-connection. A vertex-colored graph is *strong proper vertex-connected*, if for any two vertices u, v of the graph, there exists a vertex-proper u - v geodesic. For a connected graph G , the *strong proper vertex-connection number* of G , denoted by $spvc(G)$, is the smallest number of colors required to make G strong proper vertex-connected. Note that if G is a nontrivial connected graph, then

$$0 \leq pvc(G) \leq spvc(G) \leq \min\{\chi(G), srvc(G)\}. \quad (3)$$

The following results on $spvc(G)$ are immediate from definition.

Proposition 2. *Let G be a nontrivial connected graph of order n . Then*

- (a) *$spvc(G) = 0$ if and only if G is a complete graph;*
- (b) *$spvc(G) = 1$ if and only if $diam(G) = 2$.*

It is easy to obtain the following consequences.

Observation 1.

- (1) $spvc(P_3) = 1$ and $spvc(P_n) = 2$ for $n \geq 4$;
- (2) $spvc(C_4) = spvc(C_5) = 1$, $spvc(C_n) = 2$ for $n \geq 6$ even, and $spvc(C_n) = 3$ for $n \geq 7$ odd;
- (3) $spvc(K_{s,t}) = 1$ for $s \geq 2$ and $t \geq 1$;
- (4) $spvc(K_{n_1, n_2, \dots, n_k}) = 1$ for $k \geq 3$ and $(n_1, n_2, \dots, n_k) \neq (1, 1, \dots, 1)$;
- (5) $spvc(W_n) = 1$ for $n \geq 4$.

In this section, sharp upper and lower bounds of $spvc(G)$ are given for a connected graph G of order n , that is, $0 \leq spvc(G) \leq n - 2$. We also characterize the graphs of order n such that $spvc(G) = n - 2, n - 3$, respectively. Furthermore, we investigate the relationship among the three vertex-coloring parameters, namely $spvc(G)$, $srvc(G)$ and $\chi(G)$ of a connected graph G .

3.1 Bounds and characterization of extremal graphs

The problem of finding bounds of $srvc(G)$ has been solved completely by Li et al. [10].

Lemma 1. [10] *Let G be a connected graph of order n ($n \geq 3$). Then $0 \leq srvc(G) \leq n - 2$. Moreover, the bounds are sharp.*

Lemma 2. [10] *Let G be a nontrivial connected graph of order n . Then $srvc(G) = n - 2$ if and only if $G = P_n$.*

Theorem 6. *Let G be a nontrivial connected graph of order n . Then $0 \leq spvc(G) \leq n - 2$. Equality on the right-hand side is attained if and only if $G \in \{P_3, P_4\}$.*

Proof. By (3) and Lemma 1, it is obvious that $0 \leq spvc(G) \leq srvc(G) \leq n - 2$. On one hand, we know that $spvc(P_3) = 1 = n - 2$ and $spvc(P_4) = 2 = n - 2$. On the other hand, if $spvc(G) = n - 2$, then $srvc(G) = n - 2$. It follows that $G \in \{P_3, P_4\}$ from Observation 1 and Lemma 2. \square

Theorem 7. *Let G be a nontrivial connected graph of order n . Then $spvc(G) = n - 3$ if and only if G is one of the twelve graphs in Figure 1.*

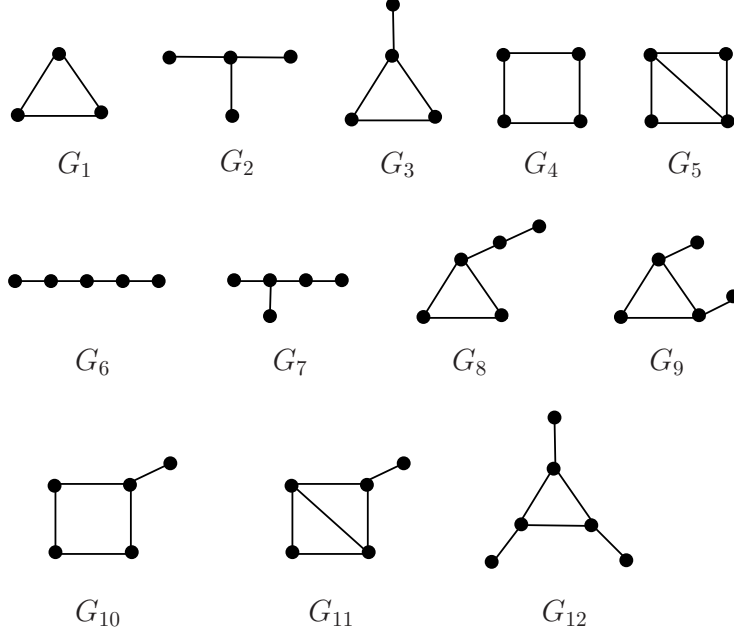


Figure 1: The twelve graphs in Theorem 7.

In order to prove Theorem 7, we need the lemma below.

Lemma 3. *If G is a connected graph with order $n \geq 7$, then $spvc(G) < n - 3$.*

Proof. Let Δ be the maximum degree of G .

Case 1. $\Delta = n - 1$. Then, we have $diam(G) \leq 2$. By Proposition 2, it follows that $spvc(G) \leq 1 < n - 3$.

Case 2. $\Delta = n - 2$. Let v be a vertex with the maximum degree Δ and $N(v) = \{v_1, v_2, \dots, v_{n-2}\}$ denote its neighborhood. Let v' be the only vertex not adjacent to v and $N(v')$ denote its neighborhood. Color the vertices of $N(v')$ with color 1 and all other vertices with color 2. Now, we will show that for any two vertices u and w in G , there exists a vertex-proper geodesic between them. If $d(u, w) \leq 2$, then there must be a vertex-proper u - w geodesic. Since G is connected, there exists a vertex v_i ($i \in \{1, 2, \dots, n-2\}$) such that v' is adjacent to v_i . If $d(u, w) = 3$ for $u = v', w \in N(v)$ (or $u \in N(v), w = v'$), then they are connected by a vertex-proper geodesic $uv_i v w$ (or $u v v_i w$). For the other cases, $d(u, w) \leq 2$. Therefore, $spvc(G) \leq 2 < n - 3$.

Case 3. $\Delta = n - 3$. Let v be a vertex with the maximum degree Δ and $N(v) = \{v_1, v_2, \dots, v_{n-3}\}$ denote its neighborhood. Let v' and v'' be the vertices not adjacent to v . Moreover, let $N(v')$ and $N(v'')$ denote the neighborhood of v' and v'' , respectively.

Subcase 3.1. $N(v') \cap N(v'') \neq \emptyset$. Then, there exists one vertex v_i ($i \in \{1, 2, \dots, n-3\}$) such that v' and v'' are adjacent to v_i . Color the vertex v_i with color 1 and all the other

vertices with color 2. Next, we shall show that there exists a vertex-proper geodesic between any two vertices u and w in G . If $d(u, w) = 3$ for $u = v'$ (or v'') and $w \in N(v)$, then they are connected by a vertex-proper path uv_ivw . If $d(u, w) = 3$ for $u \in N(v)$ and $w = v'$ (or v''), then there is a vertex-proper path uvv_iw between them. For the other cases, $d(u, w) \leq 2$. Therefore, $spvc(G) \leq 2 < n - 3$.

Subcase 3.2. $N(v') \cap N(v'') = \emptyset$. Color the vertices of $N(v')$ with color 1, the vertices of $N(v'')$ with color 2 and all the others with color 3. Firstly, suppose that $v'v''$ is an edge of G . If $N(v'') = \{v'\}$, then there exists a vertex v_i ($i \in \{1, 2, \dots, n - 3\}$) such that v' is adjacent to v_i . By a similar discussion of Case 2, we obtain that $G - v''$ is strong proper vertex-connected. Furthermore, the color of v' which is the unique neighbor of v'' , is distinct from others. Thus, there exists a vertex-proper geodesic between any two vertices in G . Similarly, the case that $|N(v'')| \geq 2$ can be proved. Now, assume that v' is not adjacent to v'' . If there is an edge between $N(v')$ and $N(v'')$, say v_1v_2 with $v_1 \in N(v')$ and $v_2 \in N(v'')$, then $v'v_1v_2v''$ is a vertex-proper $v'-v''$ geodesic; otherwise, $v'v_1vv_2v''$ is a vertex-proper $v'-v''$ geodesic. It can be verified for any other pair of vertices in G that there exists a vertex-proper geodesic between them. Hence, $spvc(G) \leq 3 < n - 3$.

Case 4. $\Delta \leq n - 4$. By (3), we have $spvc(G) \leq \chi(G)$. If G is an odd cycle, then $\chi(G) = 3$, and so $spvc(G) \leq 3 < n - 3$; otherwise, $\chi(G) \leq \Delta$ by Brook's theorem [2], and so $spvc(G) \leq \Delta < n - 3$.

The proof is complete. \square

Now, we are ready to prove Theorem 7.

Proof of Theorem 7. By Proposition 2, we obtain that $spvc(G) = n - 3$ for $G = G_i$ ($1 \leq i \leq 5$). If $G = G_i$ ($6 \leq i \leq 11$), then $diam(G) \geq 3$, and so $spvc(G) \geq 2$. A 2-coloring of the vertices of $G = G_i$ ($6 \leq i \leq 11$) is shown in Figure 2 to make G strong proper vertex-connected. Thus, $spvc(G) = n - 3$ for $G = G_i$ ($6 \leq i \leq 11$). For the graph G_{12} , color the three non-leaves with distinct colors. Then, we can see that there exists a vertex-proper geodesic for any two vertices. Hence, $spvc(G_{12}) \leq 3$. However, if one colors the vertices of G_{12} with two colors, there exist two non-leaves having the same color and then we can not find a vertex-proper geodesic between the corresponding pendant vertices.

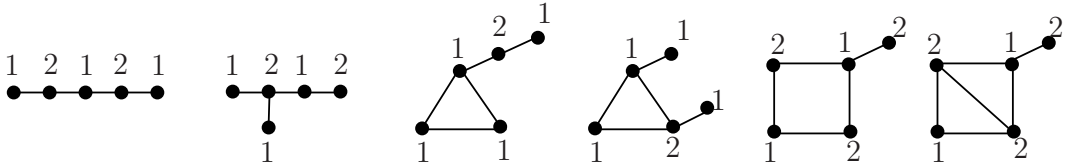


Figure 2: A 2-coloring of vertices of $G = G_i$ ($6 \leq i \leq 11$).

It remains to verify the converse. Let G be a connected graph of order $n \geq 3$ such that $spvc(G) = n - 3$. By Lemma 3, we have that $n \in \{3, 4, 5, 6\}$. Firstly, suppose $n = 6$. Then, $spvc(G) = 3$. By Proposition 2, we get $diam(G) \geq 3$. Moreover, G contains a cycle; otherwise, G is a tree and $spvc(G) = 2$. If $G \neq G_{12}$, it follows that G contains a subgraph isomorphic to one of the graphs $H_1, H_2, H_3, H_4, H_5, H_6$ in Figure 3, where a minimum vertex-coloring is also shown for each graph to make it strong proper vertex-connected. Moreover, color the remaining vertices of G with any used color if there exist. It is easy to check that G is strong proper vertex-connected. Thus, $spvc(G) \leq 2 < 3$, which is a contradiction. If $n = 5$, then $spvc(G) = 2$. By Proposition 2, $diam(G) \geq 3$. However, for $G \neq G_i$ ($6 \leq i \leq 11$), the diameter of G is at most two, which is a contradiction. Similarly, we deduce that $G = G_i$ ($2 \leq i \leq 5$) for $n = 4$ and $G = G_1$ for $n = 3$. \square

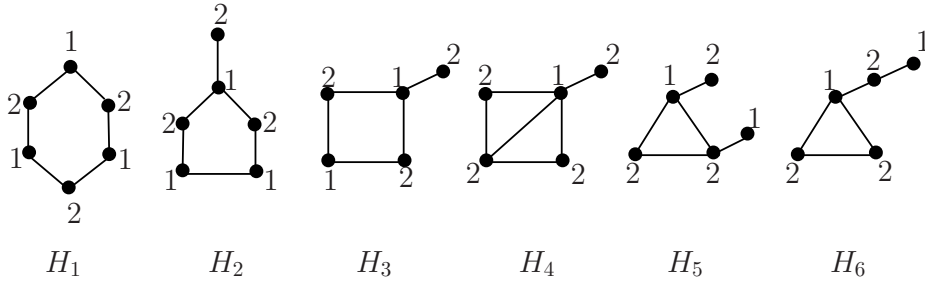


Figure 3: Subgraphs H_1, H_2, \dots, H_6 in the proof of Theorem 7.

3.2 Relationship of $spvc(G)$, $srvc(G)$ and $\chi(G)$

By (3), if G is a nontrivial connected graph with $diam(G) \geq 3$ such that $spvc(G) = a$ and $srvc(G) = b$, then $2 \leq a \leq b$. Actually, this is the only restriction on the two parameters.

Theorem 8. *For every pair a, b of integers where $2 \leq a \leq b$, there exists a connected graph G such that $spvc(G) = a$ and $srvc(G) = b$.*

Proof. Let H be the corona $cor(K_a)$ of the complete graph K_a with $V(K_a) = \{v_1, v_2, \dots, v_a\}$ and $V(H \setminus K_a) = \{v'_1, v'_2, \dots, v'_a\}$, where v'_i is the corresponding pendant vertex of v_i for $1 \leq i \leq a$. Let $F = P_{b-a}$ with $V(F) = \{w_1, w_2, \dots, w_{b-a}\}$. Now let G be the graph obtained from H and F by adding the edge $v'_a w_1$.

Firstly, we show that $spvc(G) = a$. Define a vertex-coloring of G as follows: Assign the color j to v_j for $1 \leq j \leq a$ and the color 1 to v'_a . For $1 \leq k \leq b - a$, if k is even, assign the color 1 to w_k ; otherwise, assign the color 2 to w_k . We can see that every two

vertices x and y are connected by a vertex-proper x - y geodesic. Hence, $spvc(G) \leq a$. If one colors the vertices of G with fewer than a colors, then there must be two vertices v_s and v_t , where $1 \leq s, t \leq a$, such that they have the same color. However, we can not find a vertex-proper geodesic between v'_s and v'_t . Thus, $spvc(G) = a$.

Next, we prove that $srvc(G) = b$. Define a vertex-coloring of G by assigning (1) the color j to v_j for $1 \leq j \leq a$, (2) the color $a + 1$ to v'_a and (3) the color $a + 1 + k$ to w_k for $1 \leq k \leq b - a - 1$. Since all non-leaves of G have distinct colors, G is strong rainbow vertex-connected. Hence, $srvc(G) \leq b$. If one colors the vertices of G with fewer than b colors, then there must be two vertices of $\{v_1, v_2, \dots, v_a, v'_a, w_1, w_2, \dots, w_{b-a-1}\}$ having the same color. Furthermore, the colors of $\{v'_a, w_1, w_2, \dots, w_{b-a-1}\}$ must be distinct since there is only one path between v_a and w_{b-a} . If the colors of v_i and v_j ($1 \leq i, j \leq a$) are the same, then there does not exist a vertex-rainbow geodesic between v'_i and v'_j . If the colors of v_i and w_k , where $1 \leq i \leq a - 1$ and $1 \leq k \leq b - a - 1$, are the same, then we can not find a vertex-rainbow geodesic between v'_i and w_{k+1} . If the colors of v_a and w_k ($1 \leq k \leq b - a - 1$) are the same, then there does not exist a vertex-rainbow geodesic between v_1 and w_{k+1} . Thus $srvc(G) = b$. \square

We saw in (3) that if G is a nontrivial connected graph with $diam(G) \geq 3$ which is not an odd cycle such that $spvc(G) = a$, $\chi(G) = b$ and $\Delta(G) = c$, then $2 \leq a \leq b \leq c$. In fact, this is the only restriction on the three parameters.

Theorem 9. *For every triple a, b, c of integers where $2 \leq a \leq b \leq c$, there exists a connected graph G such that $spvc(G) = a$, $\chi(G) = b$ and $\Delta(G) = c$.*

Proof. Let $H = K_b$ with $V(K_b) = \{v_1, v_2, \dots, v_b\}$. Then, add $c - b + 1$ pendant vertices, denoted by $\{v_1^1, v_1^2, \dots, v_1^{c-b+1}\}$, to v_1 , and a pendant vertex v_i^1 to v_i for $2 \leq i \leq a$. Write G as the resulting graph. It is easy to see that the maximum degree of G is c , i.e., $\Delta(G) = c$.

In the following, we first show that $spvc(G) = a$. Define a vertex-coloring of G by assigning the color j to v_j for $1 \leq j \leq a$. It is easy to check that every two vertices x and y are connected by a vertex-proper x - y geodesic. Hence, $spvc(G) \leq a$. If one colors the vertices of G with fewer than a colors, then there must be two vertices v_j and v_k ($1 \leq j, k \leq a$) such that they have the same color. However, we can not find a vertex-proper geodesic between v_j^1 and v_k^1 . Thus $spvc(G) = a$.

Next, we show that $\chi(G) = b$. Define a vertex-coloring of G by assigning (1) the color j to v_j ($1 \leq j \leq b$), (2) the color $j - 1$ to v_j^1 ($2 \leq j \leq a$) and (3) the color 2 to v_1^k ($1 \leq k \leq c - b + 1$). We can see that any two adjacent vertices have distinct colors. Hence, $\chi(G) \leq b$. If one colors the vertices of G with fewer than b colors, then there must exist two adjacent vertices v_j and v_k ($1 \leq j, k \leq b$) such that they have the same color. Thus $\chi(G) = b$. \square

References

- [1] E. Andrews, E. Laforge, C. Lumduanhom, P. Zhang, On proper-path colorings in graphs, *J. Combin. Math. Combin. Comput.*, to appear.
- [2] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.
- [3] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, Z. Tuza, Proper connection of graphs, *Discrete Math.* **312(17)** (2012) 2550-2560.
- [4] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, *Math. Bohemica* **133** (2008) 85-98.
- [5] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, The rainbow connectivity of a graph, *Networks* **54(2)** (2009) 75-81.
- [6] L. Chen, X. Li, M. Liu, Nordhaus-Gassum-type theorem for the rainbow vertex-connection number of a graph, *Util. Math.* **86** (2011) 335-340.
- [7] L. Chen, X. Li, Y. Shi, The complexity of determining the rainbow vertex-connection of a graph, *Theoret. Comput. Sci.* **412(35)** (2011) 4531-4535.
- [8] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, *J. Graph Theory* **63** (2010) 185-191.
- [9] E. Laforge, C. Lumduanhom, P. Zhang, Characterizations of graphs having large proper connection numbers, to appear in *Discuss. Math. Graph Theory*.
- [10] X. Li, Y. Mao, Y. Shi, The strong rainbow vertex-connection of graphs, *Util. Math.* **93** (2014) 213-223.
- [11] X. Li, Y. Shi, On the rainbow vertex-connection, *Discuss. Math. Graph Theory* **33** (2013) 307-313.
- [12] X. Li, Y. Shi, Y. Sun, Rainbow connections of graphs: A survey, *Graphs Combin.* **29(1)** (2013) 1-38.
- [13] X. Li, Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, New York, 2012.
- [14] H. Liu, Â. Mestre, T. Sousa, Rainbow vertex k -connection in graphs, *Discrete Appl. Math.* **162** (2013) 2549-2555.